Fundación BBVA

Documentos de Trabajo

7

2008

Carmen Herrero

Managing Waiting Lists in a Fair Way

Fundación BBVA

Managing Waiting Lists in a Fair Way

Carmen Herrero

UNIVERSITY OF ALICANTE
INSTITUTO VALENCIANO DE INVESTIGACIONES ECONÓMICAS (Ivie)

Abstract

The situation of patients on a waiting list and the problem of managing it is an example of the simplest allocation problem: the allocation of indivisible units of a homogeneous good among a set of agents demanding one unit of the good each. This working paper studies this situation, in both a static and dynamic framework, from an axiomatic perspective.

Key words

Waiting lists, axiomatic procedure, impartiality, monotonicity, seniority.

Resumen

La situación a la que se enfrentan los pacientes en lista de espera y el problema de su gestión es un ejemplo sencillo de asignación de recursos: la asignación de unidades de un bien indivisible entre un conjunto de agentes que demandan, cada uno, una unidad del bien. En este documento de trabajo se estudia esta situación, tanto en un contexto estático como dinámico, desde una perspectiva axiomática.

Palabras clave

Listas de espera, procedimiento axiomático, imparcialidad, monotonía, tiempo de espera.

Managing Waiting Lists in a Fair Way

EDITA / PUBLISHED BY
Fundación BBVA, 2008
Plaza de San Nicolás, 4. 48005 Bilbao

DEPÓSITO LEGAL / LEGAL DEPOSIT NO.: M-30.564-2008
IMPRIME / PRINTED BY: Rógar, S. A.

Impreso en España – *Printed in Spain*

La serie Documentos de Trabajo de la Fundación BBVA está elaborada con papel 100% reciclado, fabricado a partir de fibras celulósicas recuperadas (papel usado) y no de celulosa virgen, cumpliendo los estándares medioambientales exigidos por la legislación vigente.

The Working Papers series of the BBVA Foundation is produced with 100% recycled paper made from recovered cellulose fibre (used paper) rather than virgin cellulose, in conformity with the environmental standards required by current legislation.

El proceso de producción de este papel se ha realizado conforme a las normas y disposiciones medioambientales europeas y ha merecido los distintivos Nordic Swan y Ángel Azul.

The paper production process complies with European environmental laws and regulations, and has both Nordic Swan and Blue Angel accreditation.

C O N T E N T S

1. Introduction

THE simplest rationing problem is, most likely, one in which there is a certain supply of a homogeneous good, coming in indivisible units, and a set of agents demanding one unit of the good each, where the number of agents is larger than the available units of the good. Different examples of this situation are the college admission problem, when the number of students is larger than the places available, the allocation of grants among students, the demand for tickets for a performance, the problem of the demobilization of soldiers in a war, or the case of waiting lists for surgical operations.

In many cases, what we face is a *static* problem: at a single point in time both demand and supply of a certain good appear, and this situation is one-shot as, for example, in the college admission problem. Here, we should simply identify the subset of agents who will enjoy one unit of the good each. In other circumstances, nonetheless, there are different time periods in which supply and demand for the good are generated, and agents that were not served at one period may still demand the good in a future period, as in the case of surgical operations, vacancies in old peoples' homes, etc. In this case, then, we face a *dynamic* problem, and solving it means identifying, at any point in time, the subset of agents that should enjoy the good. In both cases we are interested not in solving a single problem, but in getting a general procedure, or *rule,* to solve any problem. Furthermore, we would like our rules to fulfil *procedural or ethical properties,* that is, we want to apply an *axiomatic method.*

We can model the previous situation in the following way: agents come to a counter to ask for one unit of the good, and they are registered at the counter. At a time t, we have the set of agents N_t registered at the counter. Now, a planner considers the agents in N_t, and then he decides which of them, S_t, should be served at time t. At time $t+1$ a new set of agents demanding the good, N_{t+1}, is under consideration, and the planner decides which of them, S_{t+1}, should be served, and so on. Notice, nonetheless, that in the dynamic setting, if $(N_t \setminus S_t) \cap N_{t+1} \neq \emptyset$, some interconnections should appear in the way the planner decides on the sets of agents to be served within N_t and N_{t+1}, respectively.

An easy way of solving the previous problems is by using the principle of *first come*, *first served.* According to this principle, agents are registered by their arrival order, and the planner just keeps this order to serve the agents, as in the supply of tickets for a theater performance. Nonetheless, this principle is not always sound in ordering the agents when there is a shortfall in supply. For instance, in the college admission problem, the qualifications of the students plays an important role, or in the management of waiting lists for medical attention, considerations of urgency, health deterioration, etc. are generally taken into account. That is, sometimes there are characteristics of the agents, other than their arrival order, that should be considered in order to decide who should be served first.

By considering all possible combinations of the relevant characteristics for the particular problem at hand, we have, instead of agents, *labels* or *types* that can be attached to real agents when they come to the counter. What we can do is, instead of considering the problem of ordering the actual set of agents we face at any point in time, just to order the *types* we face. A planner is *impartial* if he does not look at the agents themselves, but only takes into account their types when ordering them.

The consideration of types gives rise, normally, to multiple allocations. The idea of an *allocation criterion* (instead of allocation rule) responds to this fact. Again, we are not interested in solving (even by a multiplicity of allocations) a single problem, but in obtaining appropriate procedures to solve any problem.

Finally, out of an allocation criterion, we would like to select a single allocation, again under general circumstances, and in doing so fulfill the nicest properties possible. And this should be done both in the static and in the dynamic cases.

In this working paper we analyze these types of situations, and select, both in the static and intertemporal settings, allocation rules and allocation criteria fulfilling a set of natural properties. Basically, our selected rules function according to different linear orders on the set of agents (either those present in the problem at hand, or the set of all potential agents). Allocation criteria come from a linear order on the set of types, not enough to fully select the set of agents to be served. Thus, what we should do in this case is to effectively select the set of agents to be served by using as subsidiary a preestablished order on the set of agents.

The dynamic case also considers properties of stationarity and seniority, namely, some sort of time consistency and priority of those agents going from one period to the next. This implies that the set of types should always be ordered in the same way, across time, and that agents should be subsidiary, ordered by their arrival time.

In the particular case of hospital waiting lists there is a general consensus on the two aspects that should be taken into account when managing the lists: *clinical priority and date in the list* (see Hador and The Steering Committe of the Western Canada Waiting List Project, 2000; Walterturnbull, 2005; Norfolk and Norwich University Hospital Patient Access, 2005; Mart, 2002). Also stressed is the *need for reevaluation of patients in the list* since their clinical situation may vary while waiting (see Cochrane, 1999; Kelly, 2001; Bernal, 2002). When looking for theoretical approaches to the problem, nonetheless, there are not many references. Young (1994) presents a theoretical model for the static case, but no theoretical model exists, to the best of our knowledge, addressing the dynamic situation.

The working paper is structured as follows. In section 2 we analyze the static case, and in section 3 the intertemporal framework. An example of ordering types by means of a *point system* is considered in section 4, followed by comments and final remarks in section 5.

2. Formal Model I: At a Point in Time

2.1. Allocation rules

There is an infinite set of potential agents, $\boldsymbol{N}$. Let us call *FS* the family of all finite subsets of *N*. Let us call $\mathbb{N}$ the set of posive integer numbers. Time is measured in discrete units, $t = 1, 2, ...$ In this section, we consider the time period fixed. For the sake of simplicity, from now on, we will omit the subscript *t*. Consequently, we are now at a static framework.

A *problem* is a pair $(N, s) \in FS \times \mathbb{N}$, such that $\# N \geq s$, where *N* is interpreted as the set of agents who demand the good, and *s* is the number of available units of the good, or, alternatively, the number of agents that can be served. Let *P* be the set of all possible problems. We face the problem of selecting, for any problem, the set of agents that should be served. This idea amounts to defining the concept of *solution* or *allocation rule.*

Allocation rule. This is a function $F: P \rightarrow FS$ such that two conditions are fulfilled:

$$F(N, s) \subset N, \tag{2.1}$$

and

$$\# F(N, s) = s. \tag{2.2}$$

That is, a solution specifies, for any possible problem, how to allocate all the available units of the good among the agents present in the problem. The first property we would like an allocation rule to fulfill is the following: Suppose that for a particular problem (N, s), the rule prescribes that some group of agents *A* should be served. Suppose now that the available units of the good decrease, namely, instead of the problem (N, s), we face the problem (N, s'), with $s' < s$. Then, in the new problem we should serve a subset of the previously served agents, namely, $F(N, s') \subset A$. The fact that now there is less amount of the good available should not favour those agents not previously served. We shall call this property *resources monotonicity,* in line with similar properties in other contexts, the common idea being that, under a change in the available resources, all agents should be affected in the same direction: if resources increase, nobody should be worse off, or, symmetrically, if resources decrease, nobody should be better off. This principle has been extensively applied to different allocation models (Curiel, Maschler and Tijs, 1988; Young, 1987; Thomson, 1995; etc.).

Resources monotonicity. For any two problems (N, s), $(N, s') \in P$, with the same set of agents, if $s' < s$, then $F(N, s') \subset F(N, s)$.

Supose now that, for any finite set of agents, $N \in F$, we have a linear order $\succ_N$ on N, namely, $\succ_N$ is a complete, irreflexive and transitive relation on N. Call $\succ_N(N) = (a_1, a_2, \ldots, a_n)$ a permutation of the elements in N, ordered according to $\succ_N$, that is, $a_1 \succ_N a_2 \succ_N \ldots \succ_N a_n$. Now, for any $N \in FS$ and for any integer number $s > 0$, let us call $s(N, \succ_N) = \{a_1, \ldots, a_s\}$ the set of the s first elements in $\succ_N(N)$. Then, we can define a rule in the following way: For all $(N, s) \in P$, $F(N, s) = s(N, \succ_N)$. Rule F simply assigns the available units of the good to the s first agents, according to the order $\succ_N$. It is immediate to check that rule F satisfies resources monotonicity. The next result says that if we want a rule to satisfy resources monotonicity, we have to serve the agents in every N, according to some linear order.

Theorem 1. *An allocation rule F satisfies resources monotonicity if for all $N \in FS$ there is a strict order $\succ_N$ on the set N, such that $F(N, s) = s(N, \succ_N)$ for all $(N, s) \in P$.*

Proof. It suffices to prove the *if* part. Let F be a rule satisfying resources monotonicity, and let N be a set of agents. Consider the following order $\succ_N$ defined on N: $a_1 = F(N, 1)$, and let $a_i = F(N, i) - F(N, i-1)$. Since F satisfies resources monotonicity, $a_1 \succ_N a_2 \succ_N \ldots \succ_N a_{i-1} \succ_N a_i \succ_N \ldots$ is a linear order on N. Now, by construction, $F(N, s) = s(N, \succ_N)$.

Now consider two principles, involving changes in the set of agents. The first one, *consistency,* says that solving subproblems should be done according to the way we solve larger problems. The second one, *population monotonicity,* says that the arrival of new agents without an increase in the resources should not favor those agents present in the first problem that were not served before. Again, these principles have been extensively applied to allocation problems (see Thomson, 1995, 1996).

Consistency. For all $(N, s) \in P$, for all $A \subset N$, $F(A, k) = A \cap F(N, s)$, where $k = \#[A \cap F(N, s)]$.

Population monotonicity. For all *(N, s), (T, s)* $\in$ *P,* such that $N \subset T$, $F(T, s) \cap N \subset F(N, s)$.

Now, we have the following result:

Theorem 2. *An allocation rule F is either consistent or population monotonic if there is a strict order* $\succ$ *on the set* ***N*** *of potential agents such that for all* $(N, s) \in P$, $F(N, s) = s(N, \succ_N)$, *where* $\succ_N$ *stands for the restriction of* $\succ$ *to the set N.*

Proof. Obviously, whenever there is a strict order to the set of potential agents, it induces a strict order in any subset *N*. Furthermore, if we define $F(N, s) = s(N, \succ_N)$, and now consider a subset $A \subset N$, on the set *A*, then $\succ_A$ coincides with the restriction of $\succ_N$ to the set *A*, and thus, *F* satisfies both consistency and population monotonicity.

Reciprocally, let *F* be an allocation rule, and $a, b \in \mathbf{N}$. Define $a \succ b \Leftrightarrow F(\{a, b\}, 1) = \{a\}$. We see that when *F* is either consistent or population monotonic, the order $\succ$ defined this way is a strict order on the set of potential agents ***N***. In fact, it is enough to prove that $\succ$ is transitive. Let $a, b, c, \in \mathbf{N}$ such that $a \succ b$, $b \succ c$. This means that $F(\{a, b\}, 1) = \{a\}$; $F(\{b, c\}, 1) = \{b\}$.

Suppose first that *F* is consistent, and consider the problem $(\{a, b, c\}, 2)$. It cannot be that $F(\{a, b, c\}, 2) = \{a, c\}$, since it would imply, by consistency, that $F(\{b, c\}, 1) = \{c\}$, against the hypothesis. Similarly, it cannot be that $F(\{a, b, c\}, 2) = \{b, c\}$, since again by consistency, $F(\{a, b\}, 1) = \{b\}$. Thus, $F(\{a, b, c\}, 2) = \{a, b\}$, and again, by consistency, $F(\{a, c\}, 1) = \{a\}$, i.e., $a \succ c$.

Now, assume instead that *F* is population monotonic, and consider the problem $(\{a, b, c\}, 1)$. It cannot be that $F(\{a, b, c\}, 1) = \{b\}$, since it would imply, by pop-monotonicity, that $F(\{a, b\}, 1) = \{b\}$. It cannot be, either, that $F(\{a, b, c\}, 1) = \{c\}$, because it would imply, again by pop-monotonicity, that $F(\{b, c\}, 1) = \{c\}$. Consequently, $F(\{a, b, c\}, 1) = \{a\}$. Thus, by population monotonicity, $F(\{a, c\}, 1) = \{a\}$, i.e., $a \succ c$.

Now it can be seen that for all $(N, s) \in P$, $F(N, s) = s(N, \succ_N)$, where $\succ_N$ stands for the restriction of $\succ$ to the set *N*.

Theorem 2 states the equivalence between *consistency* and *population monotonicity* for these sorts of allocation rules, apart from characterizing the rules satisfying these properties. Furthermore, it states that consistent (and therefore population-monotonic) allocation rules also satisfy resources monotonicity, a less demanding property.

2.2. Types. Priority. Allocation criteria

An obvious way of ordering the agents in any particular problem is by their arrival order. But taking only the arrival order into account is not always a sound way of serving the agents. In many cases, some characteristics of the agents are considered as relevant for allocation purposes. For example, in the college admission problem, it could be relevant to take into account the qualifications of the students in several fields, the secondary school they come from, etc. In the allocations of grants, apart from previous information, the family income also could be relevant. In the health care provision problem we may consider urgency, health deterioration, likelihood of recovery, age and comorbidity, etc. Those characteristics allow us to classify the agents into *types*. Let Υ be the set of types. Consider now the function $\tau: \boldsymbol{N} \rightarrow \Upsilon$, so that $\tau\ (a)$ is the type of agent a. We shall assume that τ is exhaustive, namely, for any type $\tau\ \in\ \Upsilon,\ \exists\ \ a\ \in\ \boldsymbol{N}$ such that $\tau\ (a) = \tau$. By classifying the agents into types, we avoid considerations other than those relevant to the assignment problem. Namely, if two agents, a, b are such that $\tau\ (a) = \tau\ (b)$, we may interpret that a and b, if present in a problem, are indistinguishable to the planner, *according to their relevant characteristics*.

Now, the classification of agents into types is done because *not all types should have an equal right to be served*. Consequently, a way of *comparing* types in this respect is called for. Namely, we have to define a relation ρ for the set of types Υ, so that for all $\tau, \tau' \in \Upsilon$, $\tau\rho\tau'$ if and only if an agent of type τ has at least as much right to the good as an agent of type τ'. Relation ρ, normally known as a *priority relation*, is complete and transitive, namely, it is a weak order on the set of types Υ. Of course, ρ also induces a weak order on the set of agents in the obvious way: $a\rho b \Leftrightarrow \tau\ (a)\,\rho\tau\ (b)$. Since relation ρ induces only a weak order on the set of agents, it is not enough to provide them with a linear order. Obviously, out of the priority relation ρ, we can construct the *strict priority relation* $\tilde{\rho}$, so that $\tau\tilde{\rho}\tau' \Leftrightarrow \tau\rho\tau'$ and not $\tau'\ \rho\tau$. Analogously, the *similar priority relation* $\bar{\rho}$, so that $\tau\bar{\rho}\tau' \Leftrightarrow \tau\rho\tau'$ and $\tau'\rho\tau$. Whenever two types $\tau, \tau' \in \Upsilon$ are such that $\tau\tilde{\rho}\tau' \Leftrightarrow$ type τ has a *strict priority* over τ'. If $\tau\bar{\rho}\tau'$, for any tipes τ and τ', we say that they *are in a par.*

A priority relation on the set of types can help us design an allocation rule, but it is not enough to do so. When we use a priority relation ρ to build up a rule, it is because we want our rule to assign the available units of the good to those agents with the highest priority. As a consequence, if we face a problem (N, s), and it turns out that by only looking at the agents′ types we can identify exactly s agents $S \subset N$, so that if $a \in S$, and $b \in N \backslash S$, it always happens that $\tau(a)\,\tilde{\rho}\tau\ (b)$, then our prescription should be $F\,(N, s) = S$.

But if we have too many agents in a par, then the priority relation ρ by itself is not enough to completely describe the allocation rule. In such cases, ρ only can identify a set of different allocations, all of them *equally fair,* but is unable to unambiguously select one of the aformentioned allocations. Consequently, what we can do, instead of constructing an allocation rule, is to construct what is normally called an *allocation criterion.*

Allocation criterion. This is a set valued mapping $\Phi: P \to\to FS$ such that for all $A \in \Phi\ (N, s)$,

$$A \subset N, \tag{2.3}$$

and

$$\# A = s. \tag{2.4}$$

An allocation criterion selects, for every problem, the set of *fair allocations* for such a problem. As a consequence, $\Phi\ (N, s)$, does not identify the set of agents to be served, but it identifies several sets of possible groups of agents to be served, all of them equally fair *according* to the types chosen.

Whenever we have a priority relation ρ, we can define an allocation criterion by selecting, for any problem $(N, s) \in P$, all allocations with exactly s individuals each, all of them with higher (or equal) priority than any of those left out of that allocation. Namely, we can define Φ^{ρ}, as follows: for all $(N, s) \in P$, $\Phi^{\rho}\ (N, s) = \{A \subset N: \# A = s,$ and $a \rho b$, for all $a \in A$, all $b \in N \setminus A\}$.

Now, we shall consider properties our allocation criteria should satisfy. The first one corresponds to the idea of *impartiality:* what matters is the type, and not the name, of any agent involved in a problem. An allocation criterion Φ is said to be *anonymous* if it selects fair allocations according only to the relevant data (type) and not to the names of the agents.

Anonymity. Let (N, s), $(N', s) \in P$ such that $\# N = \# N'$, and there exists a one-to-one function $\theta: N \to N'$, with $\tau\ (\theta\ (a)) = \tau\ (a)$, for all $a \in N$. Then, for all $A \in \Phi\ (N, s)$, we have that $\theta\ (A) \in \Phi\ (N', s)$.

An allocation criterion selects, for any problem, a group of fair allocations. If an individual belongs to all of those allocations, he will be served for sure. If not, the relative number of fair allocations to which he belongs somehow measures his *relative right* to be served, in that particular problem, according to the allocation criterion. For an allocation criterion Φ, for any $(N, s) \in P$, and for any agent $a \in N$, call $\alpha^{\Phi}\ [a; (N, s)]$ the number of allocations in $\Phi\ (N, s)$ containing a over the total number of allocations in $\Phi\ (N, s)$.

Relative right. Of agent $a \in N$, in problem (N, s) and allocation criterion Φ,

$$\alpha^{\Phi}[a; (N, s)] = \frac{\# \{A \in \Phi\ (N, s): a \in A\}}{\# \{A \in \Phi\ (N, s)\}}.$$

Suppose that an individual leaves the problem. It would not be fair that, because of that individual leaving, any of the remaining agents strictly decreases his relative right. On the contrary, it should be fair that the relative right of all remaining individuals weakly increases. An allocation criterion is said to be *monotonic with respect to the population* if whenever an individual disappears from a problem, everybody else's right weakly increases. Notice that, again, we use the principle of population monotonicity in this context under its general meaning, namely, if an agent disappears from a problem, nobody should suffer; simmetrically, if a new agent appears in a problem, nobody should benefit.

Population monotonicity. Let $(N, s), (N', s) \in P$ such that $N' \subset N$. Then, for all $b \in N'$,

$$\alpha^{\Phi}\ [b; (N', s)] \geq \alpha^{\Phi}\ [b; (N, s)].$$

It is immediate to check that the allocation criterion constructed out of a priority relation, Φ^{ρ}, is anonymous and pop-monotonic. The next result says that those are the only allocation criteria fulfilling these properties:

Theorem 3. *An allocation criterion Φ is anonymous and population-monotonic if there is a priority relation ρ on the set of types, Υ, such that $\Phi\ (N, s) = \Phi^{\rho}\ (N, s)$, for all $(N, s) \in P$.*

Proof. Let Φ be an allocation criterion fulfilling anonymity and population monotonicity. Because of anonymity, all problems involving the same group of types are solved identically. Construct now the following relation on Υ: for all $\tau, \tau' \in \Upsilon$, $\tau \rho \tau' \Leftrightarrow \exists\ a,b \in N$, such that $\tau\ (a) = \tau$, $\tau\ (b) = \tau'$, and $\Phi\ (\{a, b\}, 1) = \{a\}$ or either $\Phi\ (\{a, b\}, 1) = \{\{a\}, \{b\}\}$.

Let us prove that ρ is a priority relation. For all τ, $\tau' \in \Upsilon$, $\exists\ a, b \in \mathbf{N}$ such that $\tau\ (a) = \tau$, and $\tau\ (b) = \tau'$. Then, we have the following possibilities: (a) $\Phi\ (\{a, b\}, 1) = \{a\}$, and thus $\tau \rho \tau'$, (b) $\Phi\ (\{a, b\}, 1) = \{\{a\}, \{b\}\}$, and $\tau \rho \tau'$, or (c) $\Phi\ (\{a, b\}, 1) = \{b\}$, and thus $\tau' \rho \tau$. By anonymity, the same holds true independently of the agents chosen. So, ρ is complete.

Now, let $\tau, \tau', \tau'' \in \Upsilon$ be such that $\tau \rho \tau'$ and $\tau' \rho \tau''$. Let $a, b, c, \in \mathbf{N}$ such that $\tau\ (a) = \tau$, $\tau\ (b) = \tau'$, and $\tau\ (c) = \tau''$. Suppose that it is not true that $\tau \rho \tau''$, namely, $\Phi\ (\{a, c\}, 1) = \{c\}$. Then, by pop-monotonicity, $\Phi\ (\{a, b, c\}, 1)$ only has two possibilities, it is either $\Phi\ (\{a, b, c\}, 1) = \{c\}$, or else, $\Phi\ (\{a, b, c\}, 1) = \{b\}$.

In the first case, and again, by pop-monotonicity, Φ $(\{c, b\}, 1) = \{c\}$. But then it is not true that $\tau'\rho\tau''$, against the hypothesis. In the second case, Φ $(\{a, b\}, 1) = \{b\}$, contradicting $\tau\rho\tau'$. Consequently, ρ is transitive, and therefore it is a priority relation. It is straightforward to check that $\Phi = \Phi^{\rho}$.

Another proceduraly interesting property for allocation criteria is pairwise consistency (Young, 1994). It informally says that when distributing a single unit among two agents it has to be done according to the way we distribute a larger number of units in a larger set. Formally,

Pairwise consistency. For all $a, b, \in \boldsymbol{N}$, all $A \in F$ such that $\{a, b\} \subset A$,

$$\text{for all } C \in \Phi \ (A, 1), \text{ with } C \cap \{a, b\}, \neq \emptyset \ C \in \Phi \ (\{a, b\}, 1), \tag{2.5}$$

and

$$\text{for all } D \in \Phi \ (\{a, b\}, 1), \text{ all } C \in \Phi \ (A, 1), \text{ with } C \cap \{a, b\} \neq \emptyset, (C \setminus \{a, b\}) \cup D \in \Phi \ (A, 1). \tag{2.6}$$

It is immediate to check that the allocation criterion constructed out of a priority relation, Φ^{ρ}, is anonymous and pairwise consistent. The next result says that those are the only allocation criteria fulfilling these properties:

Theorem 4 (Young, 1994). *An allocation criterion Φ is anonymous and pairwise consistent if there is a priority relation ρ on the set of types, Υ, such that Φ $(N, s) = \Phi^{\rho}$ (N, s), for all $(N, s) \in P$.*

Proof. Let Φ be an allocation criterion fulfilling anonymity and pairwise consistency. Because of anonymity, all problems involving the same group of types are solved identically. Construct now the following relation on Υ: for all $\tau, \tau', \in \Upsilon$, $\tau\rho\tau' \Leftrightarrow \exists\, a, b \in \boldsymbol{N}$, such that $\tau(a) = \tau, \tau\ (b) = \tau'$ and Φ $(\{a, b\}, 1) = \{a\}$ or either Φ $(\{a, b\}, 1) = \{\{a\}, \{b\}\}$.

It was proven in theorem 2 that this relation is complete, because of anonymity. We now see that it is also transitive. Let $\tau, \tau', \tau'' \in \Upsilon$ be such that $\tau\rho\tau'$ and $\tau'\rho\tau''$. Let $a, b, c \in \boldsymbol{N}$ such that $\tau(a) = \tau, \tau\ (b) = \tau'$, and $\tau\ (c) = \tau''$. Suppose that it is not true that $\tau\rho\tau''$, namely, Φ $(\{a, c\}, 1) = \{c\}$. Consider the problem $(\{a, b, c\}, 1)$. It cannot be that $\{a\} \in \Phi$ $(\{a, b, c\}, 1)$, since in such a case, pairwise consistency will imply $\{a\} \in \Phi$ $(\{a, c\}, 1)$. Suppose that $\{b\} \in \Phi$ $(\{a, b, c\}, 1)$. Then, by pairwise consistency, $\{b\} \in \Phi$ $(\{a, b\}, 1)$. But in this case, also $\{a\} \in \Phi$ $(\{a, b\}, 1)$, and then $\{a\} \in \Phi$ $(\{a, b, c\}, 1)$. Suppose, finally, that $\{c\} \in \Phi$ $(\{a, b, c\}, 1)$. Then, by pairwise consistency, $\{c\} \in \Phi$ $(\{b, c\}, 1)$. But then also $\{b\} \in \Phi$ $(\{b, c\}, 1)$, and then $\{b\} \in \Phi$ $(\{a, b, c\}, 1)$. Contradiction. Consequently, ρ is transitive.

Theorems 3 and 4 show that there is an important parallel between pairwaise consistent allocation criteria, population monotonic allocation criteria, and rational choice, in the presence of impartiality. That is, if an impartial allocation criterion treats agents consistently or respecting population monotonicity, then pairwise decisions can be rationalized by an ordering. Under impartiality, population monotonicity and pairwise consistency are equivalent.

2.3. Allocation rules compatible with an allocation criterion

Given an allocation criterion Φ, an *allocation rule compatible with* Φ is nothing but a *selection of* Φ, namely, a function $F: P \to FS$ such that for all $(N, s) \in P$, $F(N, s) \in \Phi(N, s)$. To construct an allocation rule compatible with an allocation criterion we need a breaking ties rule, in order to choose one of the fair allocations prescribed by Φ, whenever $\Phi(N, s)$ is multivalued. We may choose any way of breaking ties, as for instance, by coin toss, or any other probabilistic device. Alternatively, we may choose any preestablished linear order on the agents in N: Passport number, date of birth, name, or more in line with the idea of rights, in the case of waiting lists, arrival order.

Let (N, s) be a problem such that $\Phi(N, s)$ is multivalued. Then, we have two types of agents: those with a full right to be served, namely those $a \in N$ such that $\alpha^{\Phi}[a; (N, s)] = 1$, and those who do not have a full right to be served, namely those $b \in N$ with $\alpha^{\Phi}[b; (N, s)] < 1$. Thus, $N = A \cup B$, where $A = \{a \in N: \alpha^{\Phi}[a; (N, s)] = 1\}$, and $B = \{b \in N: \alpha^{\Phi}[b; (N, s)] < 1\}$. Then, we have the following result:

Proposition. *If* Φ *is an anonymous and population-monotonic allocation criterion, then, for all* $b, b' \in B$, $\alpha^{\Phi t}[b; (N_t, s_t)] = \alpha^{\Phi t}[b'; (N_t, s_t)]$.

Proof. By theorem 3, if Φ is anonymous and population-monotonic, there is a priority relation ρ, such that $\Phi = \Phi^{\rho}$. Let $b, b' \in B$. Then $\tau(b), \bar{\rho}\tau(b')$, and consequently, for all $A \in \Phi(N, s)$ such that $b \in A$, there exists another $A' \in \Phi(N, s)$ such that $b' \in A'$. Consequently,

$$\neq \{A \in \Phi(N, s): b \in A\} = \neq \{A \in \Phi(N, s): b' \in A\},$$

and thus,

$$d^{\Phi t}[b; (N_t, s_t)] = d^{\Phi t}[b'; (N_t, s_t)].$$

Now, suppose that we have an order in $N, \succ_N$, and let us order the agents in B according to $\succ_N$, namely, take $\succ_N (B) = (b_1 \succ_N b_2 \succ_N \ldots \succ_N b_K)$, and let be $C_i = (b1, \ldots, b_i)$ the set of the *ith* first agents in B according to $\succ_N$. If $\neq A = u$, then we choose the allocation: $A \cup C_{s_t} - u$. This amounts to defining the following allocation rule:

$$F^{\Phi}_{\succ N}(N, s) = \begin{cases} \Phi_t \ (N_t, s_t) \ \# \ \Phi_t \ (N_t, s_t) = 1 \\ A \cup C_{s_t} - u \ \# \ \Phi_t \ (N_t, s_t) \succ 1 \end{cases}.$$

$F^{\Phi}_{\succ N}$, is, by construction, a *resources-monotonic* allocation rule *compatible with Φ*. The result, nonetheless, is stronger. Any *resources-monotonic* allocation rule compatible with Φ is of the previous type.

Furthermore, if, instead of some specific order for any set, we consider a linear order $\succ$ on the set of potential agents, $\mathbf{N}$, and, for any particular agents set, N, the restriction of order $\succ$ to the set N, we get an allocation rule compatible with Φ that also satisfies consistency and population monotonicity. The next results say that this is the only way of selecting allocation rules out of anonymous and population-monotonic (or pairwise consistent) allocation criteria fulfilling the desired properties.

Theorem 5. *Let Φ be an anonymous and pop-monotonic allocation criterion, and let F be an allocation rule compatible with Φ. Then, F satisfies resources monotonicity if for all $N \in F$ there is a strict order $\succ_N$ on the set N, such that $F(N, s) = F^{\Phi}_{\succ N}(N, s)$, for all $(N, s) \in P$.*

Proof. It follows directly from theorems 1 and 3.

Suppose now that we have a linear order $\succ$ in the set of potential agents, $\mathbf{N}$, and, for any problem $(N, s) \in P$, we take $F(N, s) = F^{\Phi}_{\succ N}(N, s)$, where $\succ_N$ stands for the restriction of $\succ$ to the set N. Such an allocation rule is consistent and population-monotonic. Furthermore, those are the only rules fulfilling those properties compatible with Φ.

Theorem 6: *Let Φ be an anonymous and population-monotonic allocation criterion, and let F be an allocation rule compatible with Φ. Then, F is consistent if there exists a strict order $\succ$ in the set of potential agents* **N**, *such that $F(N, s) = F^{\Phi}_{\succ N}(N, s)$, for all $(N, s) \in P$, where $\succ_N$ stands for the restriction of $\succ$ to the set N.*

Proof. It follows from theorems 2 and 3.

3. Formal Model II: Intertemporal Problems

LET us consider now a stream of problems, one arising at any period of time. Assume, furthermore, that we consider a certain time horizon, namely, we would like to solve all possible problems arising from $t = 1$ up to $t = T$. An *intertemporal problem* is, then, a sequence of problems, each of them dated at a specific period t, namely, $\{(N_t, s_t)\}_{t=1}^{T} \in FS \times \mathbb{N}^T$. Let P^T be the set of all possible intertemporal problems. We also look now for solutions to solve any intertemporal problem. This gives rise to the concept of *allocation rules* in this intertemporal setting.

Allocation rule. This is a function $F: \rho^T \to F^T$, $F\ [\{(N_t, s_t)\}_{t=1}^{T}] = \Pi_{t=1}^{T} F_t\ (N_t, s_t)$, such that for all $t = 1, \ldots, T$,

$$F_t\ (N_t, s_t) \subset N_t, \tag{3.1}$$

and

$$\#F_t\ (N_t, s_t) = s_t. \tag{3.2}$$

An allocation rule, then, specifies how to allocate the available amount of the good at any time in any particular stream. Consequently, any of the components F_t are nothing but allocation rules at t each. Now we can extend the idea of *resources monotonicity* for intertemporal problems.

Resources monotonicity. For any problem $\{(N_t, s_t)\}_{t=1}^{T}$, for any t^*, if $F_{t^*}\ (N_{t^*}, s_{t^*}) = S_{t^*}$, and $\{(N'_t, s'_t)\}_{t=1}^{T}$ is such that $(N_t, s_t) = (N'_t, s'_t)$, $t < t^*$, $N_{t^*} = N'_{t^*}$, and $s'_{t^*} < s_{t^*}$, then $F_{t^*}\ (N'_{t^*}, s'_{t^*}) \subset S_{t^*}$.

In other words, F is resources-monotonic if all its components, F_t are resources-monotonic. We then know (by theorem 1) that F satisfies resources-monotonicity if for every problem $\{(N_t, s_t)\}_{t=1}^{T}$, every t, and every N_t a linear order on N_t, $\succ_{N_t}$ exists so that $F_t\ (N_t, s_t) = s_t\ (N_t, \succ_{N_t})$.

Similarly, population monotonicity and consistency can be considered in this setting.

Population monotonicity. For any pair of problems $\{(N_t, s_t)\}_{t=1}^{T}$, and $\{(N'_t, s'_t)\}_{t=1}^{T}$, for any t^*, if $(N_t, s_t) = (N'_t, s'_t)$, $t < t^*$, $N_{t^*} \subset N'_{t^*}$, and $s_{t^*} = s'_{t^*}$, then $F_{t^*} (N'_{t^*}, s_{t^*}) \cap N_{t^*} \subset F_{t^*} (N_{t^*}, s_{t^*})$.

Consistency. For any pair of problems $\{(N_t, s_t)\}_{t=1}^{T}$, and $\{(N'_t, s'_t)\}_{t=1}^{T}$, for any t^*, if $(N_t, s_t) = (N'_t, s'_t)$, $t < t^*$, $N_{t^*} \subset N'_{t^*}$, and $s'_{t^*} = \# [N_{t^*} \cap F_{t^*} (N_{t^*}, s_{t^*})]$, then $F_{t^*} (N'_{t^*}, s_{t^*}) \cap N_{t^*} \subset F_{t^*} (N_{t^*}, s_{t^*})$.

Again, an intertemporal allocation rule is population monotonic and/or consistent if all its components are, respectively, population monotonic and/or consistent. Because of theorem 2, this happens if for every t, a linear order on the set of potential agents $\boldsymbol{N}$ exists, $\succ_t$ so that $F_t (N_t, s_t) = s_t (N_t, \succ_t)$.

Furthermore, and if for all $t = 1, ..., T$, we have a *set of types at t*, Υ, we can define the idea of *allocation criteria.*

Allocation criterion. This is a set-valued mapping $\Phi : \rho^T \to\to F^T$, $\Phi [\{(N_t, s_t)\}_{t=1}^{T}] = \prod_{t=1}^{T} \Phi_t (N_t, s_t)$ such that for all t, and all $A_t \in \Phi_t (N_t, s_t)$,

$$A_t \subset N_t, \tag{3.3}$$

and

$$\# A_t = s_t. \tag{3.4}$$

Namely, an allocation criterion $\Phi [\{(N_t, s_t)\}_{t=1}^{T}] = \prod_{t=1}^{T} \Phi_t (N_t, s_t)$ selects the family of fair allocations for that stream. Consequently, the components of Φ, Φ_t $t = 1, ..., T$, should be nothing but allocation criteria at t, for $t = 1, 2, ..., T$, each.

The anonymity of an intertemporal allocation criterion is defined so that each of its components are anonymous, i.e., they select the agent according to their type at any t, and not according to the agents' names.

Given an allocation criterion Φ, any problem $P = \{(N_t, s_t)\}_{t=1}^{T}$, and any agent $a \in N$, we can define the *relative right of agent a in problem P, at time t, and allocation criterion* Φ:

$$\alpha_t^{\Phi} [a; P] = \frac{\# \{A \in \Phi_t (N_t, s_t) : a \in A\}}{\# \{A \in \Phi_t (N_t, s_t)\}} \; .$$

Now, population monotonicity of an intertemporal allocation criterion is defined so that all its components are population-monotonic. We then know (by theorem 2), that if we want an allocation criterion Φ to be anonymous and population-monotonic, there are priority relations ρ_t on Υ_t, so that $\Phi_t (N_t, s_t) = \Phi_t^{Pt} (N_t, s_t)$.

The reason to consider a problem over time is twofold:

I) since we are dealing with scarcity situations, some individuals will be present at subsequent problems in time, and it seems fair to take this fact into account;
II) because of previous fact, some sort of stationarity in the way of solving subsequent problems is required.

First, notice that since the set of types at any t is made by selecting some characteristics of the agents that are considered relevant for allocation purposes, it seems reasonable to keep it constant across time, namely, $\Upsilon_t = \Upsilon$, for all t. Note, nonetheless, that the type of a particular agent may change across time, so it is important then to consider $\tau_t : N \rightarrow \Upsilon$.

Suppose that we keep the set of types fixed, Υ, across time, and let a, b, be two agents such that, within some stream, $\{(N_t, s_t)\}_{t=1}^{T}$, are present (i.e., demanding one unit of the good each), both at time t and at time $t + 1$, namely $a, b \in N_t \cap N_{t+1}$. Suppose furthermore that agents a and b keep their types constant from t to $t + 1$. Then, it seems reasonable to ask their relative right to keep their order relationship, i.e., if at time t the relative right of agent a is larger than that of agent b, the same should happen at time $t + 1$.

Stationarity. For any problem $\{(N_t, s_t)\}_{t=1}^{T} \in \rho^T$ for any t^*, for any pair of agents, a, b, if $a, b \in N_{t^*} \cap N_{t^*+1}$, and $\tau_{t^*}(a) = \tau_{t^*+1}(a), \tau_{t^*}(b) = \tau_{t^*+1}(b)$, then $\alpha^{\Phi_{t^*}} [a; (N_{t^*}, s_{t^*})] < \alpha^{\Phi_{t^*}} [b; (N_{t^*}, s_{t^*})]$ iff $\alpha^{\Phi_{t^*+1}} [a; (N_t^{*+1}, s_{t^*+1})] < \alpha^{\Phi_{t^*+1}} [b; (N_{t^*+1}, s_{t^*+1})]$.

The following result says that if we want an allocation criterion to satisfy stationarity, we should keep the priority relation constant across time.

Theorem 7. *An anonymous and population-monotonic allocation criterion Φ is stationary if there is a priority relation ρ on the set of types, Υ, such that $\Phi_t = \Phi^{\rho}$, for all $t = 1, ..., T$.*

Proof. Since Φ is anonymous and population-monotonic, for all $t = 1, ..., T$ there exist priority relations ρ_t such that $\Phi_t = \Phi^{\rho_t}$. Suppose that we have stationarity. Then, we have to see that $\rho_t = \rho_{t'}$ for any two values of t. Suppose that $\rho_2 \neq \rho_1$. Then, there exist two types, τ, τ', such that $\tau \rho_1 \tau'$, and $\tau' \tilde{\rho}_2 \tau$. Let a, b be two agents such that $\tau_1(a) = \tau_2(a) = \tau$, $\tau_1(b) = \tau_2(b) = \tau'$. Now, we can construct a problem such that $N_1 = \{a, b, c\}$; $s_1 = 1$, $N_2 = \{a, b\}$, $s_2 = 1$, where

$\tau_1(c) = \tau$. Since $F_1 = \Phi^{\rho_1}$, α^{Φ_1} $[a; (N_1, s_1)] = \alpha^{\Phi_1}$ $[c; (N_1, s_1)] \geq \alpha^{\Phi_1}$ $[b; (N_1, s_1)]$. Since $F_2 = \Phi^{\rho_2}$, α^{Φ_2} $[a; (N_2, s_2)] < \alpha^{\Phi_2}$ $[b; (N_2, s_2)]$, against stationarity. Thus, $\rho_1 = \rho_2$. In a similar way we get that $\rho_1 = ... = \rho_T$.

Finally, we may think of a sensible way of selecting allocation rules out of a particular allocation criterion, across time. By theorems 5 and 7, we know that if Φ is an anonymous, population-monotonic and stationary allocation criterion, and F is an allocation rule compatible with Φ, then, F satisfies resources monotonicity if for all t, and for all $N_t \in F$ there is a strict order $\succ_{N_t}$ on the set N_t, such that $F_t\,(N_t, s_t) = F^{\Phi}_{\succ N_t}\,(N_t, s_t)$, for all $\{(N_t, s_t)\}_{t=1}^{T} \in \rho^T$.

Similarly, by theorems 6 and 7, we know that if Φ is an anonymous, population-monotonic and stationary allocation criterion, and F is an allocation rule compatible with Φ, then F is consistent if for all t, there exists a strict order $\succ_t$ on the set of potential agents $\boldsymbol{N}$, such that $F_t\,(N_t, s_t) = F^{\Phi}_{\succ t}\,(N_t, s_t)$, for all $\{(N_t, s_t)\}_{t=1}^{T} \in \rho^T$.

Let $\{(N_t, s_t)\}_{t=1}^{T}$ be a problem, and let t^* be a point in time, t^*. Call A the set of agents that, being present at time t^*, were not served at t^*, and are still present at $t^* + 1$, waiting to be served, namely, $A = N_{t^*+1} \cap N_{t^*}$. Suppose, furthermore, that $s_{t^*+1} \leq \# A$, and for all $a \in A$, $b \in N_{t^*+1} \setminus A$, $\tau(a)\rho\tau(b)$. Then, agents in A should be served first at $t^* + 1$. This idea amounts to defining allocation rules fulfilling *seniority:*

Seniority. For any problem $\{(N_t, s_t)\}_{t=1}^{T} \in \rho^T$, for any t^*, if $A = N_{t^*+1} \cap N_{t^*}$, $s_{t^*+1} \leq \# A$, and for all $a \in A$, $b \in N_{t^*+1} \setminus A$, $\tau(a)\rho\tau(b)$. Then, $F_{t^*+1}\,(N_{t^*+1}, s_{t^*+1}) \subset A$.

We may ask if, given an allocation criterion that satisfies anonymity, population monotonicity, and stationarity, it is possible to construct a consistent allocation rule compatible with the allocation criterion satisfying seniority. The answer is negative, that is, we cannot find a consistent rule compatible with the allocation criterion satisfying seniority.

Theorem 8. *There is no consistent allocation rule compatible with an anonymous, population-monotonic and stationary allocation criterion.*

Proof. Let Φ be an allocation criterion that satisfies anonymity, population monotonicity, and stationarity, and F a consistent allocation rule compatible with Φ. We know that for all t, there exists a strict order $\succ_t$ on the set of potential agents $\boldsymbol{N}$, such that $F_t\,(N_t, s_t) = F^{\Phi}_{\succ t}\,(N_t, s_t)$, for all $\{(N_t, s_t)\}_{t=1}^{T} \in \rho^T$.

Consider now a point in time, t^*, and three agents, $a, b, c \in \boldsymbol{N}$ such that $b \succ_{t^*+1} a$, and let $\{(N_t, s_t)\}_{t=1}^{T}$ be a problem such that $(N_{t^*}, s_{t^*}) = [\{a, c\}; 1]$, and $(N_{t^*+1}, s_{t^*+1}) = [\{a, b\}; 1]$, and, moreover, $\tau_{t^*}(c)\,\rho\,\tau_{t^*}(a) = \tau_{t^*+1}(a) = \tau_{t^*+1}(b)$. In such a case, we have that $F_{t^*}(N_{t^*}, s_{t^*}) = \{c\}$, while $F_{t^*+1}(N_{t^*+1}, s_{t^*+1}) = \{b\}$, against seniority.

Theorem 8 says that we cannot have consistent allocation rules, under previous circumstances, fulfilling seniority. Nonetheless, it is possible to have resources monotonic allocation rules fulfilling seniority. For any problem $\{(N_t, s_t)\}_{t=1}^{T} \in \rho^T$, and for any agent $a \in N = \cup_{t=1}^{T} N_t$, denote $t(a)$ the first period in which agent a appears in the problem, i.e, $t(a) = 1$ if $a \in N_1$; $t(a) = 2$ if $a \notin N_1$, but $a \in N_2$, and so on. Now, define a linear order on the set of agents N, $\succ_N$ such that for all $a, b \in N$, if $t(a) < t(b) \Rightarrow a \succ_N b$. Now, consider the allocation rule F such that $F_t(N_t, s_t) = F^{\Phi}_{\succ N}(N_t, s_t)$. This rule satisfies seniority by construction. Furthermore, those are the only resources-monotonic allocation rules fulfilling seniority.

Theorem 9. *A resources-monotonic allocation rule F compatible with a stationary, resources-monotonic and consistent allocation criterion Φ fulfils seniority if for any problem $\{(N_t, s_t)\}_{t=1}^{T}$ there exists an order $\succ$ on the set $N = \cup_{t=1}^{T} N_t$, such that if $t(a) < t(b) \Rightarrow a \succ_N b$, and for all $t = 1, ..., T$, $F_t(N_t, s_t) = F^{\Phi}_{\succ N}(N_t, s_t)$.*

Proof. We know that for all t, there is a linear order, $\succ_{N_t}$, so that $F_t(N_t, s_t) = F^{\Phi}_{t, \succ t}(N_t, s_t)$. Consider the set $N = \cup_{t=1}^{T} N_t$, and define the order $\succ_N$ on N as follows: $t(a) < t(b) \Rightarrow a \succ_N b$; if $t(a) = t(b) = t^*$, then $a \succ_{N_{t^*}} b \Rightarrow a \succ_N b$.

It is straightforward to check that $\succ_N$ is a linear order on N, and that for all $t = 1, ..., T$, $F_t(N_t, s_t) = F^{\Phi}_{\succ N}(N_t, s_t)$.

Theorem 9 roughly says that whenever we want to keep seniority in the allocation rules compatible with anonymous, population-monotonic and stationary allocation criteria it is by limiting ourselves to allocation rules satisfying resources monotonicity, and furthermore, that, in solving any particular stream, the selected ordering should respect the *arrival order* of the agents.

4. An Example: Point Systems

THE construction of types and the way of ordering them is not a trivial question. In general, types are constructed by considering a finite set of relevant attributes, A. Then, for every attribute $\alpha \in A$, a set of linearly ordered *levels*, $L(\alpha) = \{l_j^\alpha\}_{j \in J(\alpha)}$ is considered. That is, $l_i^\alpha >_\alpha l_j^\alpha$ if level l_i^α of attribute α is considered as *more important* than level l_j^α. This order can be translated into $J(\alpha)$, so that for all $i, j \in J(\alpha)$, $i >_\alpha j$ if $l_i^\alpha >_\alpha l_j^\alpha$. A *type* is then defined by the levels of the different attributes, $\tau = (l_j^\alpha)_{\alpha \in A, j \in J(\alpha)}$. The set of types Υ can be identified with $\Pi_{\alpha \in A} J(\alpha)$, that is, a type is defined by a vector of levels, one for each attribute. Ordering types, then, is an aggregation problem that, in general, suffers from the common difficulties of aggregation procedures. Whenever some standard properties are asked to be satisfied, we may face problems of incompatibility between them (see Young, 1994, chapter 2).

A common way of ordering types in this case, or, in other words, of establishing a priority relation on Υ, is by using a *point system*. In a point system, for every attribute $\alpha \in A$, weights are attached to its levels, $w[l_j^\alpha]$, so that for every α, and every $i, j \in J(\alpha)$, $w[l_i^\alpha] \geq w[l_j^\alpha]$ if $i >_\alpha j$. Then, a number is associated to every type, by adding up the weights associated to its levels of the different attributes, namely, $W(\tau) = \Sigma_{\alpha \in A}\, w[l_j^\alpha]$.

For example, suppose that two attributes, *gender* and *age*, are considered as relevant to order the agents in a queue. Women should go before men, children (below 15 years old) should go before elderly people (over 65), and elderly people should go before people between 15 and 65. There are four customers: *a*, a 35-year-old woman; *b*, a boy of 10 years; *c*, a 68-year-old woman, and *d*, a 70-year-old man. Our first attribute, α = *age*, takes on three levels: l_1^α = *below* 15; l_2^α = *over* 65; l_3^α = *between* 15 and 65. The second attribute, β = *gender*, takes on two levels, l_1^β = *woman*, and l_2^β = *man*. A point system in this example is nothing but a set of weights, $w(l_1^\alpha), w(l_2^\alpha), w(l_3^\alpha), w(l_1^\beta), w(l_2^\beta)$, with the condition $w(l_1^\alpha) \geq w(l_2^\alpha) \geq w(l_3^\alpha); w(l_1^\beta) \geq w(l_2^\beta)$. If, for instance, we determine that $w(l_1^\alpha) = 3$, $w(l_2^\alpha) = 2$, $w(l_3^\alpha) = 1$, $w(l_1^\beta) = 2$, $w(l_2^\beta) = 1$, then $W(a) = 1$, $W(b) = 2$, $W(c) = 2$, $W(d) = 1$, namely, *b* and *c* are in a par, and should be served before *a* and *d*, who also are in a par. If, instead we determine that $w(l_1^\alpha) = 5$, $w(l_2^\alpha) = 2$, $w(l_3^\alpha) = 0$, $w(l_1^\beta) = 1$, $w(l_2^\beta) = 0$, then $W(a) = 1$, $W(b) = 5$, $W(c) = 3$, $W(d) = 2$, and the order of the agents to be served is *b, c, d, a.*

Any point system defines a priority relation on the set of types, and consequently, can be used to define anonymous, population monotonic and pairwise consistent allocation criteria. Point systems have been widely used for this purpose, as for instance, in the demobilization of american soldiers in World War II (Stouffer et al., 1949), in the assignation of organs for transplants (see Starlz et al., 1987), or in determining the order for elective surgery (Rodríguez Mínguez, Herrero and Pinto, 2004).

It is important to observe that point systems are a good way of constructing priority relations only when there are no complementarities among the various attributes, as in our example before.

Note that, if ρ is the weak order induced by a point system in Υ, we have that for all $\tau, \tau' \in \Upsilon$, $\tau\rho\tau' \Leftrightarrow W(\tau) \geq W(\tau')$, namely, W is nothing but a *utility function for* ρ. Additionally, because of the particular form of this utility function, it happens that ρ satisfies the following property:

Separability. For all $\alpha \in A$, for all levels $j(\alpha), i(\alpha) \in J(\alpha)$, and for all two level combinations for $\beta \neq \alpha$, $(j(\beta))_{\beta \neq \alpha}$, $(i(B))_{\beta \neq \alpha}$, if we consider the types τ_1, τ_2, τ_3, τ_4, such that, $\tau_1 = [i(\alpha)\ (i(\beta))_{\beta \neq \alpha}]$, $\tau_2 = [i(\alpha)\ (j(\beta))_{\beta \neq \alpha}]$, $\tau_3 = [j(\alpha)\ (i(\beta))_{\beta \neq \alpha}]$, $\tau_4 = [j(\alpha)\ (j(\beta))_{\beta \neq \alpha}]$, it happens that $\tau_1\rho\tau_2 \Leftrightarrow \tau_3\rho\tau_4$.

Conversely, any separable weak order on the set of types can be represented by a point system (Gorman, 1968). In fact, separability implies that there exist functions $W_\alpha : J(\alpha) \to \mathbb{R}$ such that $W_\alpha\ [l_i^\alpha] \geq W_\alpha\ [l_j^\alpha]$ if $i >_\alpha j$, so that $W(\tau) = \Sigma_{\alpha \in A} W_\alpha[l^\alpha(\tau)]$, but this is nothing else than to say that the functions W_α, $\alpha \in A$, define a point system. So, the following result holds:

Theorem 9. *A weak order* ρ *on the set of types* Υ *can be represented by a point system iff* ρ *is separable in the characteristics.*

5. Final Remarks

IN this working paper we explored the existence of sound procedures to allocate a scarce resource in a particular case: when agents demand just one unit of an indivisible good. This situation arises in many circumstances, as in the college admission problem or the allocation of grants, when the number of students demanding the good is larger than the positions or grants available, the allocation of tickets for a performance, community care for old people, or in the medical case, the allocation of organs for transplants, or surgical operations.

Two different situations are considered: static and dynamic. In a static problem, we simply face a particular problem, and agents that are not served do not have the opportunity of having the good later on. In the dynamic setting, nonetheless, agents keep waiting to be served at a later period of time.

A crucial element in solving the problem is the idea that *agents have different rights to be served first, according to their characteristics*. Different combinations of the relevant characteristics amount to define *types*. Allocation procedures are *impartial* whenever they only take into account the agents' types. Next, the main problem is how to order types, i.e., to define a *priority relation on the set of types*.

Once this problem is solved, we also need a breaking ties rule in order to choose a particular set of agents, in case the ordering of the types is not enough. Here, some particular order on the set of agents (unrelated to their type) is called for. In the static setting, in order to fulfill as many nice properties as possible, it is enough that the breaking ties rule is an ex ante priority relation on the set of agents.

In the dynamic setting, some sort of stationarity is required: the set of types, as well as the priority relation on them should be kept fixed across time; agents' type should be reevaluated at every period, and finally, the breaking ties rule should respect the order arrival of the agents.

Previous results (in accordance with what has been considered in practice) have been justified in the paper from an axiomatic point of view. As for the way of constructing types and priority relations on them, we also pointed out the conditions on the characteristics (separability) for a point system to properly work. Finally, let us mention that, provided separability in the characteristics hold, the most widely acceptable way of eliciting weights in point systems is by seeking opinions either from the affected parties, the general population, or from representatives of various concerned groups (medical experts, hospital administrators, patient groups, etc.). The problem here is that differences of opinion must be reconciled in order to arrive at a prioritization that represents some sort of social *consensus*. Each particular problem, thus, will need some specific way of treatment.

References

BERNAL, E. (2002): "Has the Time Arrived for the Management of Waiting Lists?", *Gaceta Sanitaria,* 16, 436-439.

COCHRANE, J. (1999): "Strategy to Reformulate Waiting Lists: New Zealand Has Some Suggestions for NHS Priority System for Elective Surgery", *British Medical Journal,* 318 (7199), 1698.

CURIEL, I. J., M. MASCHLER and S. H. TIJS (1987): "Bankruptcy Games", *Zeitschrift für Operations Research,* 31, A143-A159.

GORMAN, W. M. (1968): "Structure of Utility Functions", *Review of Economic Studies,* 35, 367-390. Reprinted in C. Blackorby and A. F. Shorrocks (1995), Collected Works, 150-181.

GUTTMAN, L. (1946): "An Approach for Quantifying Paired Comparisons and Rank Orders", *Annals of Mathematical Statistics,* 17, 144-163.

HADOR, D. and THE STEERING COMMITTEE OF THE WESTERN CANADA WAITING LIST PROJECT (2000): "Setting Priorities for Waiting Lists: Defining our Terms", *Canadian Medical Association Journal,* 163 (7), 857-860.

KELLY, K. (2001): "Change in Pain and Function while Waiting for Major Joint Arthroplasty", *The Journal of Arthroplasty,* 16 (3), 351-359.

MART, J. (2002): "Management of Surgical Waiting Lists by Health Centers and Professionals", *Gaceta Sanitaria,* 16, 440-443.

NORFOLK AND NORWICH UNIVERSITY HOSPITAL PATIENT ACCESS (2005).

RODRÍGUEZ MÍNGUEZ, E., C. HERRERO and J. L. PINTO (2004): "Using Point System in the Management of Waiting Lists: The Case of Cataracts", *Social Sciences and Medicine,* 59, 585-594.

STARLZ, T. E., T. R. HAKALA, A. TZAKIS, R. GORDON, A. STIEBER, L. MAKOWKA, J. KLIMOSKI and H. T. BAHNSON (1987): "A Multifactorial System for Equitable Selection of Cadaver Kidney Recipients", *Journal of the American Medical Association,* 257 (22), 3073-3075.

STOUFFER, S. A., E. A. SUCHMAN, L. C. DE VINEY, S. A. STAR and R. M. WILLIAMS, JR. (1949): *Studies in Social Psychology in World War II: The American Soldier,* vol. 1, *Adjustment During Army Life,* Princeton, Princeton University Press, 125.

THOMSON, W. (1995): *Population-Monotonic Allocation Rules,* in W. A. Barnett, H. Moulin, M. Salles and N. Schoffield (eds.), *Social Choice, Welfare and Ethics,* Cambridge University Press, 79-124.

— (1995): "Consistent Allocation Rules", RCER Working Paper 418, University of Rochester.

WALTERTURNBULL (2005): *Review of Hospital Waiting List Management,* Final Report, November.

YOUNG, P. (1988): "Distributive Justice in Taxation", *Journal of Economic Theory,* 48, 321-335.

— (1994): *Equity in Theory and Practice,* New Jersey, Princeton University Press.

*ABOUT THE AUTHOR**

CARMEN HERRERO holds a PhD in mathematics from the University of Valencia and is a graduate of the Complutense University of Madrid. She is currently professor of economics at the University of Alicante and also a senior researcher at the Instituto Valenciano de Investigaciones Económicas (Ivie). Her latest research refers to the application of axiomatic techniques to the analysis of distributive problems, especially justice and equity analysis. She has recently been visiting lecturer at the University of California (Davis) and at the University of Rochester, and has given courses at the University of Paris X (Nanterre) and at the Institute of Advanced Studies of Vienna. Member of the board of the Spanish Science and Technology Foundation, she is also the Spanish co-ordinator of the TMR Network Cooperation and Information, FMRX-CT96-0055, and of the National R&D project SEJ 2007-62656, as well as a member of the editorial board of the review *Social Choice and Welfare*. Among her approximately fifty publications are recent articles in *Journal of Health Economics, Economic Theory, Journal of Mathematical Psychology, Mathematical Social Sciences* and *Social Choice and Welfare*.
e-mail: carmen.herrero@ua.es

Any comments on the contents of this paper can be addressed to Carmen Herrero at carmen.herrero@ua.es.

* The author acknowledges financial support from the Spanish Science and Education Ministry SEJ 2007-62656.

Fundación BBVA

D O C U M E N T O S   D E   T R A B A J O

NÚMEROS PUBLICADOS

DT 01/04 *Métodos de solución de problemas de asignación de recursos sanitarios*
Helena Ramalhinho Dias Lourenço y Daniel Serra de la Figuera

DT 01/05 *Licensing of University Inventions: The Role of a Technology Transfer Office*
Inés Macho-Stadler, David Pérez-Castrillo y Reinhilde Veugelers

DT 02/05 *Estimating the Intensity of Price and Non-price Competition in Banking: An Application to the Spanish Case*
Santiago Carbó Valverde, Juan Fernández de Guevara Radoselovics, David Humphrey y Joaquín Maudos Villarroya

DT 03/05 *Sistemas de pensiones y fecundidad. Un enfoque de generaciones solapadas*
Gemma Abío Roig y Concepció Patxot Cardoner

DT 04/05 *Análisis de los factores de exclusión social*
Joan Subirats i Humet (Dir.), Ricard Gomà Carmona y Joaquim Brugué Torruella (Coords.)

DT 05/05 *Riesgos de exclusión social en las Comunidades Autónomas*
Joan Subirats i Humet (Dir.), Ricard Gomà Carmona y Joaquim Brugué Torruella (Coords.)

DT 06/05 *A Dynamic Stochastic Approach to Fisheries Management Assessment: An Application to some European Fisheries*
José M. Da-Rocha Álvarez y María-José Gutiérrez Huerta

DT 07/05 *The New Keynesian Monetary Model: Does it Show the Comovement between Output and Inflation in the U.S. and the Euro Area?*
Ramón María-Dolores Pedrero y Jesús Vázquez Pérez

DT 08/05 *The Relationship between Risk and Expected Return in Europe*
Ángel León Valle, Juan Nave Pineda y Gonzalo Rubio Irigoyen

DT 09/05 *License Allocation and Performance in Telecommunications Markets*
Roberto Burguet Verde

DT 10/05 *Procurement with Downward Sloping Demand: More Simple Economics*
Roberto Burguet Verde

DT 11/05 *Technological and Physical Obsolescence and the Timing of Adoption*
Ramón Caminal Echevarría

DT 01/06 *El efecto de la inmigración en las oportunidades de empleo de los trabajadores nacionales: Evidencia para España*
Raquel Carrasco Perea, Juan Francisco Jimeno Serrano y Ana Carolina Ortega Masagué

DT 02/06 *Inmigración y pensiones: ¿Qué sabemos?*
José Ignacio Conde-Ruiz, Juan Francisco Jimeno Serrano y Guadalupe Valera Blanes

DT 03/06 *A Survey Study of Factors Influencing Risk Taking Behavior in Real World Decisions under Uncertainty*
Manel Baucells Alibés y Cristina Rata

DT 04/06 *Measurement of Social Capital and Growth: An Economic Methodology*
Francisco Pérez García, Lorenzo Serrano Martínez, Vicente Montesinos Santalucía y Juan Fernández de Guevara Radoselovics

DT 05/07 *Cooperation and Cultural Transmission in a Coordination Game*
Gonzalo Olcina Vauteren y Vicente Calabuig Alcántara

DT 06/07 *The* Extended *Atkinson Family and Changes in Expenditure Distribution: Spain 1973/74 – 2003*
Francisco J. Goerlich Gisbert, María Casilda Lasso de la Vega Martínez y Ana Marta Urrutia Careaga

DT 07/07 *Análisis de la evolución de la dependencia en la tercera edad en España*
David Casado Marín

DT 08/07 *Designing Contracts for University Spin-offs*
Inés Macho-Stadler, David Pérez-Castrillo y Reinhilde Veugelers

DT 09/07 *Regional Differences in Socioeconomic Health Inequalities in Spain*
Pilar García Gómez y Ángel López Nicolás

DT 10/07 *The Evolution of Inequity in Access to Health Care in Spain: 1987-2001*
Pilar García Gómez y Ángel López Nicolás

DT 11/07 *The Economics of Credit Cards, Debit Cards and ATMs: A Survey and Some New Evidence*
Santiago Carbó-Valverde, Nadia Massoud, Francisco Rodríguez-Fernández, Anthony Saunders y Barry Scholnick

DT 12/07 *El impacto comercial de la integración europea, 1950-2000*
Luis Fernando Lanaspa Santolaria, Antonio Montañés Bernal, Marcos Sanso Frago y Fernando Sanz Gracia

DT 13/07 *Proyecciones de demanda de educación en España*
Andrés M. Alonso Fernández, Daniel Peña Sánchez de Rivera y Julio Rodríguez Puerta

DT 14/07 *Aversion to Inequality and Segregating Equilibria*
Antonio Cabrales Goitia y Antoni Calvó-Armengol

DT 15/07 *Corporate Downsizing to Rebuild Team Spirit*
Antonio Cabrales Goitia y Antoni Calvó-Armengol

DT 16/07 *Maternidad sin matrimonio: Nueva vía de formación de familias en España*
Teresa Castro Martín

DT 17/07 *Immigrant Mothers, Spanish Babies: Childbearing Patterns of Foreign Women in Spain*
Marta Roig Vila y Teresa Castro Martín

DT 18/07 *Los procesos de convergencia financiera en Europa y su relación con el ciclo económico*
José Luis Cendejas Bueno, Juan Luis del Hoyo Bernat, Jesús Guillermo Llorente Álvarez, Manuel Monjas Barroso y Carlos Rivero Rodríguez

DT 19/07 *On Capturing Rent from a Non-Renewable Resource International Monopoly: A Dynamic Game Approach*
Santiago J. Rubio Jorge

Documentos de Trabajo

7

Fundación BBVA

Gran Vía, 12
48001 Bilbao
España
Tel.: +34 94 487 52 52
Fax: +34 94 424 46 21

Paseo de Recoletos, 10
28001 Madrid
España
Tel.: +34 91 374 54 00
Fax: +34 91 374 85 22

publicaciones@fbbva.es
www.fbbva.es